

## UK Junior Mathematical Olympiad 2015

Organised by The United Kingdom Mathematics Trust

Tuesday 9th June 2015

### **RULES AND GUIDELINES : READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING**

1. Time allowed: 2 hours.
2. **The use of calculators, measuring instruments and squared paper is forbidden.**
3. All candidates must be in *School Year 8 or below* (England and Wales), *S2 or below* (Scotland), *School Year 9 or below* (Northern Ireland).
4. For questions in Section A *only the answer is required*. Enter each answer neatly in the relevant box on the Front Sheet. Do not hand in rough work. Write in blue or black pen or pencil.

For questions in Section B you must give *full written solutions*, including clear mathematical explanations as to why your method is correct.

Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Front Sheet on top.

***Do not hand in rough work.***

5. Questions A1-A10 are relatively short questions. Try to complete Section A within the first 30 minutes so as to allow well over an hour for Section B.
6. Questions B1-B6 are longer questions requiring *full written solutions*. This means that each answer must be accompanied by clear explanations and proofs. Work in rough first, then set out your final solution with clear explanations of each step.
7. These problems are meant to be challenging! Do not hurry. Try the earlier questions in each section first (they tend to be easier). Try to finish whole questions even if you are not able to do many. A good candidate will have done most of Section A and given solutions to at least two questions in Section B.
8. Answers must be FULLY SIMPLIFIED, and EXACT using symbols like  $\pi$ , fractions, or square roots if appropriate, but NOT decimal approximations.

**DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!**

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## Section A

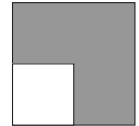
Try to complete Section A within 30 minutes or so. Only answers are required.

**A1.** It is 225 minutes until midnight. What time is it on a 24-hour digital clock?

**A2.** The diagram shows what I see when I look straight down on the top face of a non-standard cubical die. A positive integer is written on each face of the die. The numbers on every pair of opposite faces add up to 10. What is the sum of the numbers on the faces I cannot see?

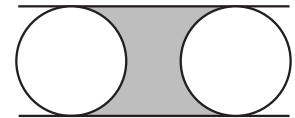


**A3.** The diagram shows one square inside another. The perimeter of the shaded region has length 24 cm. What is the area of the larger square?



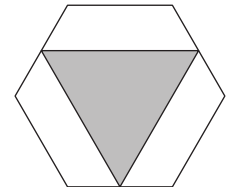
**A4.** My fruit basket contains apples and oranges. The ratio of apples to oranges in the basket is 3 : 8. When I remove one apple the ratio changes to 1 : 3. How many oranges are in the basket?

**A5.** Two circles of radius 1 cm fit exactly between two parallel lines, as shown in the diagram. The centres of the circles are 3 cm apart. What is the area of the shaded region bounded by the circles and the lines?



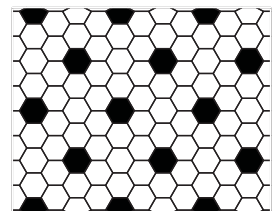
**A6.** There are 81 players taking part in a knock-out quiz tournament. Each match in the tournament involves 3 players and only the winner of the match remains in the tournament – the other two players are knocked out. How many matches are required until there is an overall winner?

**A7.** The diagram shows an equilateral triangle inside a regular hexagon that has sides of length 14 cm. The vertices of the triangle are midpoints of sides of the hexagon. What is the length of the perimeter of the triangle?

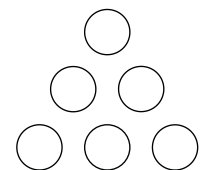


**A8.** What is the units digit in the answer to the sum  $9^{2015} + 9^{2016}$  ?

**A9.** The figure shows part of a tiling, which extends indefinitely in every direction across the whole plane. Each tile is a regular hexagon. Some of the tiles are white, the others are black. What fraction of the plane is black?



**A10.** Lucy wants to put the numbers 2, 3, 4, 5, 6 and 10 into the circles so that the products of the three numbers along each edge are the same, and as large as possible. What is this product?

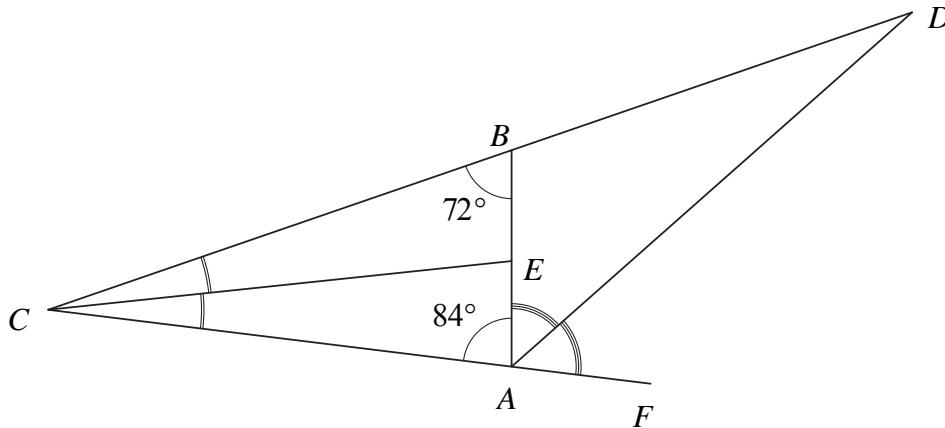


## Section B

Your solutions to Section B will have a major effect on your JMO result. Concentrate on one or two questions first and then **write out full solutions** (not just brief ‘answers’).

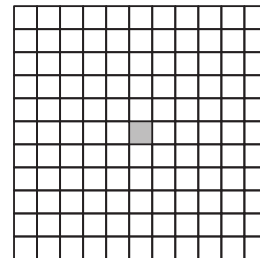
**B1.** Let  $N$  be the smallest positive integer whose digits add up to 2015. What is the sum of the digits of  $N + 1$ ?

**B2.** The diagram shows triangle  $ABC$ , in which  $\angle ABC = 72^\circ$  and  $\angle CAB = 84^\circ$ . The point  $E$  lies on  $AB$  so that  $EC$  bisects  $\angle BCA$ . The point  $F$  lies on  $CA$  extended. The point  $D$  lies on  $CB$  extended so that  $DA$  bisects  $\angle BAF$ .



Prove that  $AD = CE$ .

**B3.** Jack starts in the small square shown shaded on the grid, and makes a sequence of moves. Each move is to a neighbouring small square, where two small squares are neighbouring if they have an edge in common. He may visit a square more than once.



Jack makes four moves. In how many different small squares could Jack finish?

**B4.** The point  $F$  lies inside the regular pentagon  $ABCDE$  so that  $ABFE$  is a rhombus. Prove that  $EFC$  is a straight line.

**B5.** I have two types of square tile. One type has a side length of 1 cm and the other has a side length of 2 cm.

What is the smallest square that can be made with equal numbers of each type of tile?

**B6.** The letters  $a, b, c, d, e$  and  $f$  represent single digits and each letter represents a different digit. They satisfy the following equations:

$$a + b = d, \quad b + c = e \quad \text{and} \quad d + e = f.$$

Find all possible solutions for the values of  $a, b, c, d, e$  and  $f$ .

## UK Junior Mathematical Olympiad 2015 Solutions

**A1 20:15** The 225 minutes are equal to 3 hours and 45 minutes. At 3 hours before midnight the time is 21:00. So 45 minutes earlier than that the time is 20:15.

**A2 23** A cubical die has six faces and so three pairs of opposite faces. The numbers on each pair of opposite faces add to 10, so the sum of the numbers on all six faces is 30. Therefore the sum of the numbers on the unseen faces is  $30 - 7 = 23$ .

**A3  $36 \text{ cm}^2$**  The length of the perimeter of the shaded region is the same as the length of the perimeter of the square. Hence the square has side length 6 cm and so has area  $36 \text{ cm}^2$ .

**A4 24** Suppose there are initially  $m$  apples and  $n$  oranges in the basket. Since  $m : n = 3 : 8$ , it follows that  $8m = 3n$ . After removing one apple, the ratio becomes  $(m - 1) : n = 1 : 3$ , and so  $3(m - 1) = n$ . Putting these two equations together,

$$8m = 3 \times 3(m - 1)$$

$$8m = 9m - 9$$

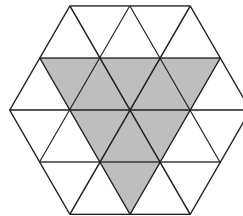
$$m = 9.$$

Using the second equation gives  $n = 3 \times 8 = 24$ . So there are 24 oranges in the basket.

**A5  $(6 - \pi) \text{ cm}^2$**  The area bounded by the circles can be calculated by subtracting the area of one circle from the area of the  $3 \text{ cm} \times 2 \text{ cm}$  rectangle with edges passing through the centres of the circle. Therefore the shaded area is  $(6 - \pi) \text{ cm}^2$ .

**A6 40** Two players are knocked out as a result of one match. To leave a winner, 80 players must be knocked out. Therefore there must be 40 matches.

**A7 63 cm** The hexagon may be divided into small equilateral triangles as shown.



Each of these small triangles has side length 7 cm and so the perimeter of the shaded triangle is  $9 \times 7 \text{ cm} = 63 \text{ cm}$ .

**A8 0** The sum can be factorised to give  $9^{2015} + 9^{2016} = 9^{2015}(1 + 9)$ . This is equal to  $9^{2015} \times 10$  and so, since this number is a multiple of 10, its units digit must be 0.

- A9**  $\frac{1}{8}$  The tiling pattern can be divided into identical pieces, each of which has the following shape.

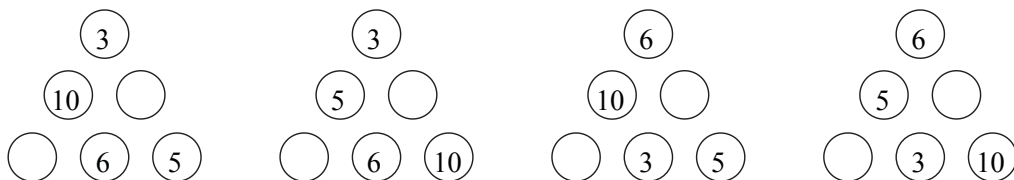


As  $\frac{1}{8}$  of each piece is black, it follows that  $\frac{1}{8}$  of the plane is black.

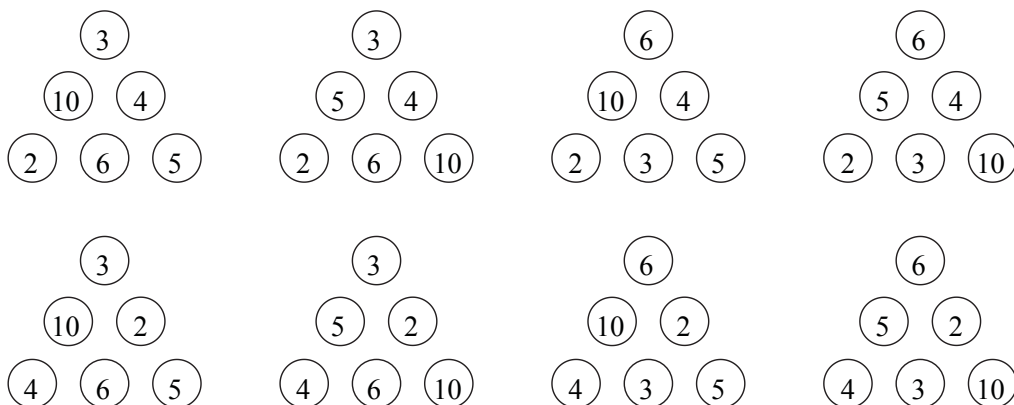
- A10 120** Each of the six numbers can be written as a product of primes as follows:

$$2, \quad 3, \quad 2 \times 2, \quad 5, \quad 2 \times 3, \quad 2 \times 5.$$

Since only two 3s appear in the list, the numbers 3 and 6 cannot appear in the same edge of the triangle; otherwise the products of the three numbers along this edge would have a factor of  $3^2$  and product of the three numbers along the other two edges would not. The same is true for 5 and 10 since there are only two 5s in the list. These observations mean that, up to symmetry, the only possible arrangements are:



For each of these arrangements, the 2 and the 4 can be placed in two ways to give the following eight arrangements.



Consider the products of the numbers along the edges of each of these arrangements. In the first arrangement, the products are all 60; in the last, the products are all 120; in each of the others, the products are not all equal. Therefore the largest possible product satisfying the conditions of the question is 120.

- B1** Let  $N$  be the smallest positive integer whose digits add up to 2015. What is the sum of the digits of  $N + 1$ ?

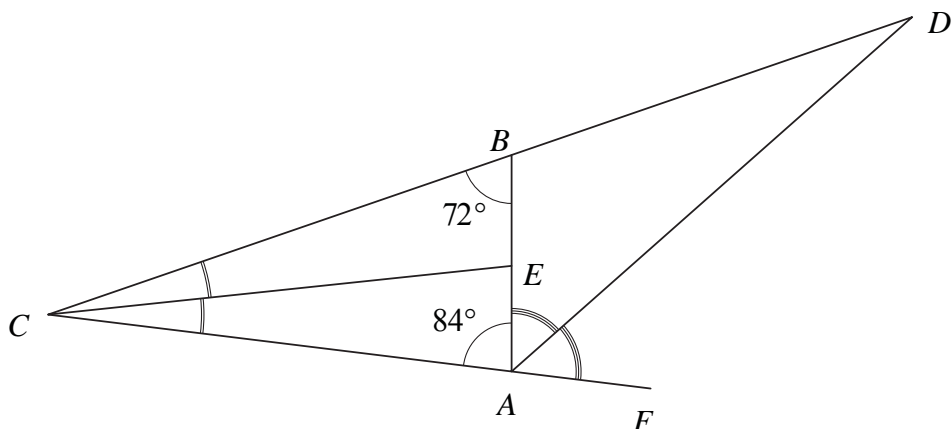
*Solution*

For such an integer to be as small as possible, it must have as few digits as possible (since any integer with more digits would be larger).

Since  $\frac{2015}{9} = 223$  remainder 8, the smallest possible number of digits is 224. Thus any number whose digits are made up of 223 copies of '9' and one '8' will have the correct digit sum and use the smallest possible number of digits. The integer  $N$  must have '8' as its leading digit, followed by 223 copies of '9'; any other arrangement of these digits would give a larger integer.

Thus  $N + 1 = 9 \times 10^{223}$ . Hence the digit sum of  $N + 1$  is 9.

- B2** The diagram shows triangle  $ABC$ , in which  $\angle ABC = 72^\circ$  and  $\angle CAB = 84^\circ$ . The point  $E$  lies on  $AB$  so that  $EC$  bisects  $\angle BCA$ . The point  $F$  lies on  $CA$  extended. The point  $D$  lies on  $CB$  extended so that  $DA$  bisects  $\angle BAF$ .



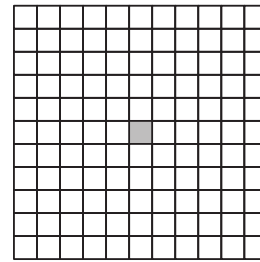
Prove that  $AD = CE$ .

*Solution*

Look first at triangle  $ABC$ . Since the angles in a triangle add to  $180^\circ$ ,  $\angle BCA = 24^\circ$ . Hence, since  $CE$  bisects  $\angle BCA$ ,  $\angle BCE = \angle ECA = 12^\circ$ . Next consider triangle  $ECA$  and use the angle sum again to obtain  $\angle AEC = 84^\circ$ . Therefore  $\angle CAE = \angle AEC$  and so triangle  $ECA$  is isosceles and hence  $AC = CE$ .

Consider the angles at  $A$ . Since angles on a straight line add to  $180^\circ$ ,  $\angle BAF = 96^\circ$ . Because  $AD$  bisects  $\angle BAF$ ,  $\angle BAD = 48^\circ$ . Finally, consider triangle  $DCA$ . Since the angles in a triangle add to  $180^\circ$ , it must be the case that  $\angle ADC = 24^\circ$ . Therefore  $\angle BCA = \angle ADC$  and so triangle  $DCA$  is isosceles and hence  $AC = AD$ . Therefore  $AD = CE$  since both are equal to  $AC$ .

- B3** Jack starts in the small square shown shaded on the grid, and makes a sequence of moves. Each move is to a neighbouring small square, where two small squares are neighbouring if they have an edge in common. He may visit a square more than once.
- Jack makes four moves. In how many different small squares could Jack finish?



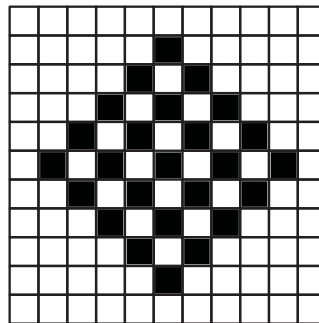
*Solution*

Let the starting square be labelled  $(0, 0)$  and the square to the right of the starting square be labelled  $(1, 0)$ , and so on for the remaining squares in the grid.

Each move may be represented by  $(+1, +0)$ ,  $(+0, +1)$ ,  $(-1, +0)$  or  $(+0, -1)$  where, for example,  $(+1, +0)$  represents a move of one square to the right and  $(+0, -1)$  represents a move of one square downwards.

Each move increases or decreases the sum of the coordinates of the occupied square by 1. At  $(0, 0)$  the sum of the coordinates is 0. After one move, the sum of the coordinates must be 1 or  $-1$ . So after two moves the sum of the coordinates is 2, 0 or  $-2$ . After three moves, the sum of the coordinates of the occupied square is 3, 1,  $-1$  or  $-3$ . Finally, after the fourth move, the sum of the coordinates of the occupied square is 4, 2, 0,  $-2$ , or  $-4$ .

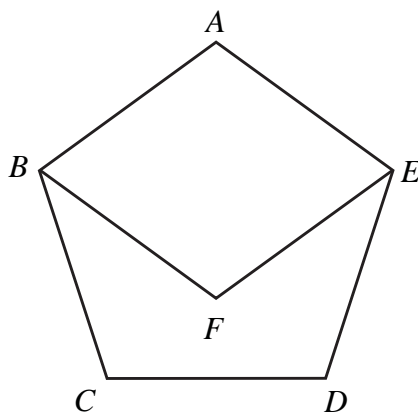
This means the sequence of four moves can end at any one of the 25 squares shown in black.



- B4** The point  $F$  lies inside the regular pentagon  $ABCDE$  so that  $ABFE$  is a rhombus. Prove that  $EFC$  is a straight line.

*Solution*

Each interior angle of a regular pentagon is  $108^\circ$ . The internal angles of a quadrilateral sum to  $360^\circ$  and so, since  $ABFE$  is a rhombus,  $\angle ABF = \angle FEA = 72^\circ$ . Therefore  $\angle FBC = 36^\circ$ . Triangle  $FBC$  is isosceles since  $BC = AB = BF$  and so  $\angle BFC = \angle BCF = 72^\circ$ . Then  $\angle EFC = \angle EFB + \angle BFC = 108^\circ + 72^\circ = 180^\circ$  and so  $EFC$  is a straight line.



- B5** I have two types of square tile. One type has a side length of 1 cm and the other has a side length of 2 cm.

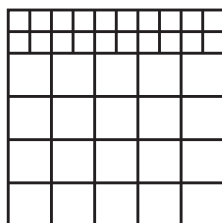
What is the smallest square that can be made with equal numbers of each type of tile?

*Solution*

Each of the 1 cm tiles has area  $1 \text{ cm}^2$  and each of the 2 cm tiles has area  $4 \text{ cm}^2$ . Suppose there are  $n$  tiles of each type. Then the assembled square must have area  $5n \text{ cm}^2$  and  $5n$  must be a square number.

The smallest  $n$  could be is 5. This would require the tiles to fit together to form a  $5 \text{ cm} \times 5 \text{ cm}$  square. However the first four 2 cm tiles are placed in a  $5 \text{ cm}$  square, the fifth 2 cm tile cannot be placed in the remaining space. So  $n$  cannot be 5.

The next smallest  $n$  for which  $5n$  is a square is 20. This would require the tiles to fit together to form a  $10 \text{ cm} \times 10 \text{ cm}$  square. This arrangement is possible; an example is shown below. Hence the smallest square that can be made is a  $10 \text{ cm} \times 10 \text{ cm}$  square.





- B6** The letters  $a, b, c, d, e$  and  $f$  represent single digits and each letter represents a different digit. They satisfy the following equations:

$$a + b = d, \quad b + c = e \quad \text{and} \quad d + e = f.$$

Find all possible solutions for the values of  $a, b, c, d, e$  and  $f$ .

*Solution*

The equations may be represented as the following triangle where each digit is the sum of the two adjacent digits directly above it.

$$\begin{array}{ccc} a & b & c \\ & d & e \\ & & f \end{array}$$

None of the digits  $a, b, c, d$  and  $e$  may be 0 since this would force two of the others to be equal. The digit  $f$  cannot be 0 since  $f = d + e$  and both  $d$  and  $e$  are positive.

Consider the middle row and note that both  $d$  and  $e$  must be at least 3 since they are each the sum of distinct positive integers. Now assume that  $d < e$ . Since  $d \geq 3$  and  $f \leq 9$  it follows that  $e \leq 6$ . This means that  $(d, e)$  is  $(3, 4), (3, 5), (3, 6)$  or  $(4, 5)$ . However, the only way to write 4 as the sum of two different nonzero digits is  $1 + 3$ ; therefore  $(d, e) \neq (3, 4)$ .

Suppose  $(d, e) = (3, 5)$ .

$$\begin{array}{ccc} a & b & c \\ & 3 & 5 \\ & & 8 \end{array}$$

Then  $b \neq 2$  since this would mean  $c = 3$ . So it must be the case that  $(a, b, c) = (2, 1, 4)$ .

Now suppose  $(d, e) = (3, 6)$ .

$$\begin{array}{ccc} a & b & c \\ & 3 & 6 \\ & & 9 \end{array}$$

Then  $a$  must be 1 or 2 and so  $(a, b, c)$  is  $(1, 2, 4)$  or  $(2, 1, 5)$ .

Finally, suppose  $(d, e) = (4, 5)$ .

$$\begin{array}{ccc} a & b & c \\ & 4 & 5 \\ & & 9 \end{array}$$

Then  $a$  must be 1 or 3. But  $a$  cannot be equal to 3 as this would force  $c$  to be 4. So  $(a, b, c) = (1, 3, 2)$ .

Therefore, with the assumption that  $d < e$ , there are four solutions. Another four can be obtained by assuming  $d > e$ , which has the effect of reflecting each of the triangles vertically.

The solutions for  $(a, b, c, d, e, f)$  are

$$\begin{array}{cccc} (2, 1, 4, 3, 5, 8), & (1, 2, 4, 3, 6, 9), & (2, 1, 5, 3, 6, 9), & (1, 3, 2, 4, 5, 9), \\ (4, 1, 2, 5, 3, 8), & (4, 2, 1, 6, 3, 9), & (5, 1, 2, 6, 3, 9), & (2, 3, 1, 5, 4, 9). \end{array}$$